

Cohomology-free diffeomorphisms on tori

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Abstract. A C^∞ diffeomorphism $\varphi: M \rightarrow M$ on a smooth closed orientable manifold M is cohomology-free (c.f.) if for each C^∞ function $f: M \rightarrow R$ there exists a constant f_0 and a C^∞ function $h: M \rightarrow R$ such that $h - h \circ \varphi = f - f_0$. In this article we show that (c.f.) diffeomorphisms on tori are conjugate to Diophantine translations. This is part of a conjecture of A. Katok [H, Problem 17]. We also show that a strictly ergodic diffeomorphism has topological entropy zero. This was a question of M. Herman [K1; 5.].

1. Introduction

We show in Proposition 4 that a (c.f.) diffeomorphism of a torus $T^n = T^p \times T^q$ may be given on the covering by

$$\varphi(x, y) = (x + \alpha, Cx + By + G(x, y) + \beta) \quad (1.1)$$

where $x \in R^p$, $y \in R^q$, $G: R^n \rightarrow R^q$ is a C^∞ \mathbb{Z}^n -periodic function and $\alpha \in R^p$ is a Diophantine vector [AS] i.e.

$$||\langle k, \alpha \rangle|| \geq \frac{C}{K^{p+\delta}}, \quad C, \delta > 0 \quad (1.2)$$

for all $k \in \mathbb{Z}^p - 0$ and $||x|| = \inf\{|x - \ell|, \ell \in \mathbb{Z}^p\}$ defines a metric on T^p where $|x| = \sup |x_j|$, $1 \leq j \leq p$ and $\langle k, \alpha \rangle = k_1\alpha_1 + \cdots + k_p\alpha_p$.

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A (c.f.) diffeomorphism leaves invariant a volume form and it is uniquely ergodic and minimal.

PROPOSITION 1. *There exists an invariant volume form for a cohomology-free diffeomorphism.*

PROOF: Let Ω_0 be a volume form on M . Thus

$$\varphi^*\Omega_0 = \det D\varphi \Omega_0 \quad (1.3)$$

since $\varphi: T^n \rightarrow T^n$ is (c.f.) then there exist a constant c and a C^∞ function $h: M \rightarrow R$ such that

$$\log |\det D\varphi| = h - h \circ \varphi + c. \quad (1.4)$$

We will show that $c = 0$. Consider the volume form

$$\Omega = \exp h \Omega_0. \quad (1.5)$$

Now from (1.4) and (1.5) we have

$$\begin{aligned} |\varphi^*\Omega| &= (\exp h \circ \varphi) |\varphi^*\Omega_0| = (\exp h \circ \varphi) |\det D\varphi| \Omega_0 \\ &= \exp(h \circ \varphi + \log |\det D\varphi|) \Omega_0 \\ &= \exp(h + c) \Omega_0 \\ &= \exp(c) \exp h \Omega_0 \end{aligned} \quad (1.6)$$

thus

$$\left| \int_M \varphi^*\Omega \right| = |\deg(\varphi)| = \exp c \int_M \Omega \quad (1.7)$$

from (1.3) and (1.4) we may assume that $\int_M \Omega = 1$ by choosing a convenient function $h: M \rightarrow R$.

Now from (1.7) we get

$$|\deg(\varphi)| = \exp(c) = 1 \quad (1.8)$$

thus $c = 0$ and from (1.5) and (1.6) we have

$$\varphi^*\Omega = \deg(\varphi)\Omega$$

□

2. The topological entropy of a strictly ergodic diffeomorphism

We say that a C^1 , diffeomorphism $\varphi: M \rightarrow M$ of a closed manifold M is *strictly ergodic* if φ is unique ergodic and the invariant measure μ is positive on the open sets. Notice that strictly ergodic diffeomorphisms are minimal. The converse is not true [He].

THEOREM A. *The topological entropy of a C^1 strictly ergodic diffeomorphism $\varphi: M \rightarrow M$ of a closed manifold M is zero.*

PROOF: Notice that the derivative $D\varphi: TM \rightarrow TM$ is an isomorphism of the tangent bundle $\pi: TM \rightarrow M$ of M linear on the fibres $D\varphi(x): T_x M \rightarrow T_{\varphi(x)} M$. Let us consider the subadditive mappings $T: M \times \mathbb{Z} \rightarrow R$ given by $T(x, n) = \|D\varphi^n(x)\|$ where $\|\cdot\|: TM \rightarrow R$ is the norm given by a Riemannian metric on M and

$$D\varphi^n(x) = D\varphi(\varphi^{n-1}(x)) \cdots D\varphi(\varphi(x)) \cdot D\varphi(x), \quad n \in \mathbb{Z}^+. \quad (2.1)$$

Now since $\log \|D\varphi(x)\| \in L^1(M, \mu)$ then the maximal Lyapunov exponent is given by the Furstenberg-Kesten Theorem [V; 3.12]

$$\lambda_+(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|D\varphi^n(x)\| \quad (2.2)$$

and since the φ -invariant measure μ is ergodic and $\lambda_+(\varphi(x)) = \lambda_+(x)$ for all $x \in M$ then λ_+ is constant.

To show that the topological entropy of φ is zero it is sufficient to show that the maximal Lyapunov exponent λ_+ is zero. We first notice that we have the subexponential decay

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|D\varphi(\varphi^j(x))\| = 0 \quad \text{for all } j \in \mathbb{Z}^+ \quad (2.3)$$

see [V; Corollary 3.11].

From (2.1) we have

$$\|D\varphi^n(\varphi(x))\| \leq \prod_{j=1}^n \|D\varphi(\varphi^j(x))\| \quad (2.4)$$

and from (2.4) we have

$$\begin{aligned} \|D\varphi^n(\varphi(x))\|^{1/n} &\leq \left(\prod_{j=1}^n \|D\varphi(\varphi^j(x))\|^{1/j} \right)^{1/n} \\ &\leq \prod_{j=1}^n \|D\varphi(\varphi^j(x))\|^{1/n} \end{aligned} \quad (2.5)$$

since $\varphi^*: C^0(M) \rightarrow C^0(M)$ is an isometry of the Banach space $C^0(M)$ of the continuous functions $f: M \rightarrow R$ with the sup norm $\|f\|_0 = \sup_{x \in M} |f(x)|$ then $\|\varphi^*\| = r(\varphi^*) = \lim_{j \rightarrow \infty} \|(\varphi^*)^j\|^{1/j} = 1$, where $r(\varphi^*)$ is the spectral radius of φ^* [L; Theorem 6.2] and from (2.5) we have

$$\begin{aligned} \frac{1}{n} \log \|D\varphi^n(\varphi(x))\| &\leq \frac{1}{n} \sum_{j=1}^n \frac{1}{j} \log \|D\varphi(\varphi^j(x))\| \\ &\leq \frac{1}{n} \sum_{j=1}^n \log \|D\varphi(\varphi^j(x))\|. \end{aligned} \quad (2.6)$$

Now

$$\frac{1}{n} \sum_{j=1}^n \frac{1}{j} C = \frac{1}{n} [\log n + \gamma + \mathcal{O}(\frac{1}{n})] C$$

where

$$C = \int_M \log \|D\varphi(x)\| d\mu(x), \quad C \geq 0$$

and

$$\frac{1}{n} < \frac{1}{n} [\log n + \gamma + \mathcal{O}(\frac{1}{n})] < 1 \quad \text{for } n > e \quad (2.7)$$

where γ is the Euler constant and $e = \sum_{k=0}^{\infty} \frac{1}{k!}$.

Taking the limit as $n \rightarrow \infty$ in (2.6) we get from (2.3) that

$$\lambda_+ = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|D\varphi^n(x)\| \leq \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \frac{1}{j} \log \|D\varphi(\varphi^j(x))\| = 0 \leq C \quad (2.8)$$

where

$$C = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log \|D\varphi(\varphi^j(x))\|.$$

uniformly by the Birkhoff Theorem since $\varphi: M \rightarrow M$ is strictly ergodic. Now from (2.8) we see that $\lambda_+ = 0$, proving the Theorem A. \square

REMARK. Unique ergodicity does not imply minimality.

PROPOSITION 2. The C^1 strictly ergodic diffeomorphisms $\varphi: M \rightarrow M$ of a closed manifold are quasi-unipotent on homology and 1 is an eigenvalue of $\varphi_*: H_1(M, R) \rightarrow H_1(M, R)$.

PROOF: By [M] and Theorem A we have the following inequality

$$\log r(\varphi_*) \leq h(\varphi) = 0$$

and from this we see that $r(\varphi_*) = 1$ where $r(\varphi_*)$ is the spectral radius of $\varphi_*: H_1(M, R) \rightarrow H_1(M, R)$. Thus by a well-known theorem in algebra all eigenvalues of φ_* are roots of unity and as $\varphi: M \rightarrow M$ is minimal then the Lefschetz fixed point theorem implies that $1 \in sp$.

PROPOSITION 3. *The (c.f) diffeomorphisms are strictly ergodic thus the topological entropies vanish.*

PROOF: The (c.f.) diffeomorphisms are strictly ergodic see [K2; Proposition 11.7] and their entropies vanish by Theorem A. \square

COROLLARY 3.1. The (c.f.) diffeomorphisms φ of a closed manifold M are quasi-unipotent on homology and 1 is an eigenvalue of $\varphi_* H_1(M, R) \rightarrow H_1(M, R)$.

PROOF: It follows from Propositions 2 and 3. \square

3. Simultaneous linearization of invariant 1-forms

PROPOSITION 4. *Let $\varphi: T^n \rightarrow T^n$ be a diffeomorphism such that $\varphi^* w_j = w_j$, $w_j = dx_j + dh_j$, $1 \leq j \leq p$. Then there exists a diffeomorphism $\psi: T^n \rightarrow T^n$ homotopic to the identity such that $\psi^* dx_j = w_j$, $1 \leq j \leq p$ and $\psi^* dy_j = dy_j$, $1 \leq j \leq q$. So if $\varphi_0 = \psi \circ \varphi \circ \psi^{-1}$ then $\varphi_0^* dx_j = dx_j$, $1 \leq j \leq p$*

PROOF: Let $\varphi: T^p \times T^q \rightarrow T^p \times T^q$, $n = p + q \in \mathbb{Z}^+$ be given on the

covering $R^n = R^p \times R^q$ by

$$\varphi(x, y) = (x + F + \alpha, Cx + By + G(x, y)) \quad (3.1)$$

where $x \in R^p$, $y \in R^q$, $F: R^n \rightarrow R^p$ and $G: R^n \rightarrow R^q$ are $C^\infty \mathbb{Z}^n$ -periodic functions. Now $\varphi^*w_j = w_j$ where $w_j = dx_j + dh_j$, $1 \leq j \leq p$ iff the cohomological equation

$$H - H \circ \varphi = F + \alpha \quad (3.2)$$

has a C^∞ solution $H: R^n \rightarrow R^p$, $H = (h_1, \dots, h_p)$. For, $\varphi^*w = w$, $w = dx + dH$ and $\varphi^*w = dx + dF + d(H \circ \varphi) = w = (dx + dH)$ iff $dH - d(H \circ \varphi) = dF$ or $H - H \circ \varphi = F + \alpha$, $\alpha \in R^p$. Let $\psi: T^n \rightarrow T^n$, $\psi(x, y) = (x + H, y)$, then ψ is a C^∞ diffeomorphism and $\psi^*dx = dx + dH = w$ and $\psi^*dy = dy$.

Thus if $\varphi_0 = \psi \circ \varphi \circ \psi^{-1}$ then φ_0 is given on the covering by

$$\varphi_0(x, y) = (x + \alpha, Cx + By + G(x, y)). \quad (3.3)$$

□

PROPOSITION 5. *The functional equation*

$$K - LK = H \quad (3.4)$$

has a unique solution $K \in C^\infty(T^n, R^q)$ for each function $H \in C^\infty(T^n, R^q)$ where $LK = BK \circ \varphi$, B is a $q \times q$ integral matrix whose spectrum $sp(B) \subset S^1 - \{1\}$ and $\varphi: T^n \rightarrow T^n$, $n = p + q$ is a minimal diffeomorphism.

PROOF: We will show that the equation (3.4) has a unique C^r solution K for each positive integer r . Let $H_t(T^n, R^q)$ be a Sobolev space [W; 6.22]. Now by the Sobolev Lemma we have

$$C^\infty(T^n, R^q) \subset H_t(T^n, R^q) \subset C^r(T^n, R^q) \quad (3.5)$$

for $t > [\frac{q}{2}] + 1 + r$ where $[\frac{q}{2}]$ denotes the greatest integer less than or equal $\frac{q}{2}$. We recall that $H_t(T^n, R^q)$ is the completion of $C^\infty(T^n, R^q)$

with respect to the norm $\|\cdot\|_t$ given locally by the bilinear function on $C^\infty(U, R^q)$, $U \subset T^q$ given by

$$(H, K)_t = \sum_{|\alpha| \leq t} \int_U \langle D^\alpha H, D^\alpha K \rangle dx \wedge dy \quad (3.6)$$

where $\alpha \in \mathbb{Z}^n$, $\alpha_i \geq 0$, $|\alpha| = \sum_{i=1}^n \alpha_i$. Notice that since $sp(B) \subset S^1 - \{1\}$ then there exists $m \in \mathbb{Z}^+$, $B^m = (I + N)$ and $N^d = 0$, $B^{m \cdot \ell} = (I + N)^\ell = (I + N)^d$ for $\ell \geq d$.

Thus there exist $C > 0$ such that $\|L^k H\|_t < C$ for all $k \in \mathbb{Z}^+$.

Now by the *Mean-Ergodic Theorem*, [L; Chap. 2,9] we have the sequence of bounded linear mappings

$$P_n = \frac{1}{n} \sum_{j=0}^{n-1} L^j \quad (3.7)$$

converging strongly to a projection P whose complementary pair of subspaces associated to the projection P are

$$\mathcal{N} = \ker T \text{ and } \mathcal{R} = CL(imT) \quad (3.8)$$

i.e. $PH = H$ for $H \in \mathcal{N}$ and $PH = 0$ for $H \in \mathcal{R}$ where $T = id - L$ and $H_t(T^n, R^q) = \mathcal{N} \oplus \mathcal{R}$. Now we claim:

$$id - P = \sigma(L)T \quad (3.9)$$

where $\sigma_n(L) = \frac{1}{n} \sum_{j=1}^n S_j(L)$, $S_j(L) = \sum_{k=0}^{j-1} L^k$ and $\sigma_n(L)$ are the Cesàro partial sums of the series $\sum_{i=0}^{\infty} L^i$ and $\lim_{n \rightarrow \infty} \sigma_n(L) = \sigma(L)$ is the Cesàro summ of the above series.

For,

$$id - L^n = (id - L)S_n(L)$$

and from this we have

$$id - \frac{1}{n} \sum_{j=1}^n L^j = \sigma_n(L)T \quad (3.10)$$

since $\sigma_n(L)$ and T commute.

Now taking the limit as $n \rightarrow \infty$ we get

$$id - P = \sigma(L)T \quad (3.11)$$

(we show at the remark below that $\mathcal{N} = \{0\}$, thus $\mathcal{R} = H_t(T^n, R^q)$). Thus $P = 0$ and $\sigma(L) = T^{-1}$. Taking $H \in \mathcal{R}$ we get

$$\begin{aligned} H &= \sigma(L)TH \\ &= \sigma(L)H - L\sigma(L)H \\ &= K - LK, \quad K = \sigma(L)H \end{aligned} \quad (3.12)$$

proving the Proposition. \square

REMARK. With the assumption of Proposition 5

$$\mathcal{N} = \ker T = \{0\} \quad \text{and} \quad P = 0.$$

PROOF: Since $sp(B) \subset S^1 - \{1\}$ there exist positive integers m and d such that

$$B^m = (id + N), \quad N^d = 0. \quad (3.13)$$

Let $H \in \mathcal{N}$ i.e. $TH = H - LH = 0$ or in matrix notation we have

$$\begin{aligned} L^m H = B^m H \circ \varphi^m &= \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_q \end{pmatrix} \circ \varphi^m + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ & \vdots & & & \vdots \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_q \end{pmatrix} \circ \varphi^m \\ &= \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_q \end{pmatrix} \end{aligned} \quad (3.14)$$

and (3.14) gives $h_1 \circ \varphi^m = h_1$ and since φ^m is minimal we see that $h_1 = c_1$, $c_1 \in \mathbb{R}$, $h_2 \circ \varphi^m - h_2 = c_1$ thus $c_1 = 0$ and $h_2 = c_2$, $c_2 \in R$, $h_3 \circ \varphi^m + c_2 = h_3$ thus $c_2 = 0$ and $h_3 = c_3, \dots, h_{q-1} = c_{q-1}$ and

$h_q \circ \varphi^m + c_{q-1} = h_q$ and we finally get $c_{q-1} = 0$ and $h_q - h_q \circ \varphi = 0$ and from this $h_q = c_q \in R$. Notice that

$$BC \circ \varphi = BC = C$$

where $C = (0, 0, \dots, 0, c_q) \in R^q$ and since $1 \notin sp(B)$ we have $C = 0$.

Thus $\mathcal{N} = \{0\}$ and from (3.11) we see that $P = 0$ and $T^{-1} = \sigma(L)$.

PROPOSITION 6. *If $\varphi: T^n \rightarrow T^n$, $n \in \mathbb{Z}^+$ is a (c.f.) diffeomorphism then there exists a C^∞ diffeomorphism $\zeta: T^n \rightarrow T^n$ homotopic to the identity conjugating φ to a skew-product given on the covering by*

$$\varphi_0(x, y) = (x + \alpha, Cx + By + S(x)) \quad (3.15)$$

where $T^n = T^p \times T^q$, $x \in R^p$, $y \in R^q$ and $S: R^p \rightarrow R^q$ is C^∞ \mathbb{Z}^p -periodic, $\alpha \in R^p$ is a Diophantine vector and B is a quasi-unipotent $q \times q$ integral matrix.

PROOF: By Corollary 3.1, φ is quasi-unipotent on homology thus by [SL; Remark 1.4] φ is given on the covering $R^n = R^p \times R^q$ by

$$\varphi(x, y) = (Ax + F(x, y), Cx + By + G(x, y)) \quad (3.16)$$

where $L = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$, $L \in GL(n, \mathbb{Z})$ and $A = I + N$, N nilpotent and C is $q \times p$ and B is $q \times q$ are integral matrices and $G(x, y)$ is a C^∞ \mathbb{Z}^n -periodic function and

$$w_j = dx_j + dh_j, \quad 1 \leq j \leq p \quad (3.17)$$

where the twisted cohomological equation

$$AH - (H \circ \varphi) + \alpha = F \quad (3.18)$$

is solved below: since $\varphi: T^n \rightarrow T^n$ is (c.f) then we have

[illegible]

and

$$\alpha_j = \int_{T^n} F_j d\mu, \quad \int_{T^n} h_j d\mu = 0, \quad 1 \leq j \leq p$$

and μ is the measure given by the φ -invariant volume form Ω .

Thus by (3.17) we have

$$\varphi^* w = {}^t A dx + d(H \circ \varphi) + dF = {}^t A w \quad (3.20)$$

in matrix notation where

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix}, \quad dx = \begin{pmatrix} dx_1 \\ \vdots \\ dx_p \end{pmatrix}, \quad H = \begin{pmatrix} h_1 \\ \vdots \\ h_p \end{pmatrix}, \quad \text{and} \quad F = \begin{pmatrix} F_1 \\ \vdots \\ F_p \end{pmatrix}.$$

To show that $\{w_1, \dots, w_p\}$ is linearly independent on T^n consider the p -form W given by

$$W = w_1 \wedge \dots \wedge w_p \quad (3.21)$$

which by (3.20) is φ -invariant.

Now $\{w_1, \dots, w_p\}$ is linearly dependent at a point $z \in T^n$ iff $W(z) = 0$ and as W is φ -invariant and φ is minimal then W vanishes on T^n giving the contradiction $dx = -d\theta$. So $\{w_1, \dots, w_p\}$ is linearly independent. We claim: the affine mapping

$$a: T^p \rightarrow T^p, \quad a(x) = Ax + \alpha \quad (3.22)$$

is a factor of φ .

For, consider the submersion $p: T^n \rightarrow T^p$ given by $p(x, y) = x + H$. Thus

$$\begin{aligned} (p \circ \varphi)(x, y) &= p(Ax + F, Cx + By + G) = Ax + F + H \circ \varphi \\ &= A(x + H) + \alpha = (a \circ p)(x, y) \end{aligned} \quad (3.23)$$

by (3.18).

Now by [SL; Corollary 1.11] the affine diffeomorphism $a: T^p \rightarrow T^p$ is (c.f.) since $\varphi: T^n \rightarrow T^n$ is (c.f.) and by [SL; Theorema 1.5] $a(x) = x + \alpha$ is a Diophantine translation.

So by [SL; Remark 1.3] and Proposition 4 there exists a diffeomorphism $\psi: T^n \rightarrow T^n$ homotopic to the identity such that $\psi^* dx_j = w_j$,

$1 \leq j \leq p$, where w_j are given in (3.17) and by [SL; Corollary 6.2] we may assume that $\psi^*(dx \wedge dy) = \Omega$. So up to a conjugation by ψ we may assume that φ is given on the covering by

$$\varphi(x, y) = (x + \alpha, Cx + By + G(x, y)) \quad (3.24)$$

and preserves the canonical volume $\Omega_0 = dx \wedge dy$ and $\varphi^*dx_j = dx_j$, $1 \leq j \leq p$. Let μ be the measure given by Ω_0 .

We claim: there exists a C^∞ diffeomorphism $\zeta: T^n \rightarrow T^n$ homotopic to the identity conjugating (3.24) to the skew-product given on the covering by

$$\varphi_0(x, y) = (x + \alpha, Cx + By + S(x)). \quad (3.25)$$

For, consider the C^∞ diffeomorphisms $\varphi_x: T^q \rightarrow T^q$ given on the covering by

$$\pi \circ \varphi(x, y) = \varphi_x(y) = Cx + By + G_x(y) \quad (3.26)$$

where

$$\pi: T^n \rightarrow T^q, \quad \pi(x, y) = y$$

is the projection, and $G_x(y) = G(x, y)$, $x \in R^p$, $y \in R^q$.

From (3.26) we get

$$\varphi_x^*dy = {}^tBdy + dG_x. \quad (3.27)$$

Now let $H_x = {}^tB^{-1}G_x$ and from (3.27) we have

$$\varphi_x^*(dy) = {}^tB(dy + dH_x) \quad (3.28)$$

as φ is minimal and $sp(B) \subset S^1 - \{1\}$ then (3.12) and Proposition 5 gives a C^∞ solution $K_x: T^q \rightarrow R^q$ to the cohomological equation

$$K_x - {}^tB^{-1}K_x \circ \varphi = H_x \quad (3.29)$$

and from (3.29) we have

$${}^tBdK_x - d(K_x \circ \varphi_x) = {}^tBdH_x. \quad (3.30)$$

For each $x \in T^p$ consider the 1-form

$$w_x = dy + dK_x \quad (3.31)$$

and from (3.28), (3.30) and (3.31) we get

$${}^t B dH_x + d(K_x \circ \varphi) = {}^t B dK_x$$

thus

$$\begin{aligned} \varphi_x^* w_x &= {}^t B(dy + dH_x) + d(K_x \circ \varphi) \\ &= {}^t B(dy + dK_x) = {}^t B w_x \end{aligned} \quad (3.32)$$

now from (3.32) and [SL; Lemma 1.6] there exists a C^∞ diffeomorphism $\psi_x: T^q \rightarrow T^q$ homotopic to the identity conjugating φ_x to the affine map

$$\psi_x \circ \varphi_x \circ \psi_x^{-1} = Cx + By + S(x) = a_x \quad (3.33)$$

so we see that $S: T^p \rightarrow R^q$ is a C^∞ \mathbb{Z}^p -periodic function. Consider the C^∞ diffeomorphism

$$\zeta: T^p \times T^q \rightarrow T^p \times T^q$$

given on the covering $R^p \times R^q$ by

$$\zeta(x, y) = (x, \psi_x(y)), \quad x \in T^p \text{ and } y \in T^q. \quad (3.34)$$

Thus

$$\varphi_0 = \zeta \circ \varphi \circ \zeta^{-1}$$

where on the covering φ_0 is given by

$$\varphi_0(x, y) = (x + \alpha, Cx + By + S(x))$$

and $\varphi_0: T^n \rightarrow T^n$ is a skew-product, $\alpha \in R^p$ is a Diophantine vector and B is quasi-unipotent, proving the Proposition 6. \square

THEOREM B. *The cohomology-free diffeomorphisms of the tori T^n , $n \in \mathbb{Z}^+$ are conjugate to Diophantine translations and the conjugating diffeomorphisms are homotopic to the identity.*

PROOF: This theorem follows from [SU; Theorem 3], Proposition 6 above and [SL; Theorem 1.5].

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